LAWRENCE-SULLIVAN MODELS FOR THE INTERVAL

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Abstract. Two constructions of a Lie model of the interval were performed by R. Lawrence and D. Sullivan. The first model uses an inductive process and the second one comes directly from solving a differential equation. They conjectured that these two models are the same. We prove this conjecture here.

This work is concerned with Lie models of the interval. Throughout this paper we assume that the base field is the field of rational numbers. A *graded Lie algebra* consists of a $\mathbb{Z}$-graded vector space L, together with a bilinear product called the *Lie bracket* that we denote $[-,-]$, such that $[x,y] = -(-1)^{|x||y|}[y,x]$ and

$$(-1)^{|x|\,|z|}[x,[y,z]] + (-1)^{|y|\,|x|}[y,[z,x]] + (-1)^{|z|\,|y|}[z,[x,y]] = 0,$$

for all homogeneous $x, y, z \in L$, where $|\alpha|$ refers to the degree of a homogeneous element $\alpha \in L$. If a graded Lie algebra is endowed with a derivation ∂ of degree -1 such that $\partial^2 = 0$, we call (L,∂) a *differential graded Lie algebra*, abbreviated *dgL*, and ∂ is its *differential*.

Let V be a $\mathbb{Z}$-graded vector space, and let TV denote the tensor algebra on V. When endowed with the commutator bracket, TV becomes a graded Lie algebra. The *free Lie algebra generated by* V, denoted $\mathbb{L}V$, is the smallest sub Lie algebra of TV containing V. An element in $\mathbb{L}V$ has *bracket length* k if it is a linear combination of iterated brackets of k elements of V, i.e., if it belongs to the intersection $\mathbb{L}V \cap T^kV$, where T^kV denotes the subspace of TV generated by the words of tensor length k. The subspace of elements of bracket length k is denoted $\mathbb{L}^kV$. If $(\mathbb{L}V,\partial)$ is a dgL,we denote by ∂_k the derivation induced by the composition of ∂ with the projection $\mathbb{L}V \to \mathbb{L}^kV$. We denote by $\widehat{\mathbb{L}}V$ the completed Lie algebra, whose elements are formal series of elements of $\mathbb{L}V$.

Let X be a CW-complex with cells (e_α) such that their closure $(\overline{e}_\alpha)$ has the rational homology of a point. Denote by V the rational vector space span by the desuspended cells, i.e., each cell e_α generates a component $\mathbb{Q}$ of V in degree $|e_\alpha|-1$. In an appendix to [6], D. Sullivan constructs a completed differential Lie algebra $(\widehat{\mathbb{L}}V,\partial)$, with $\partial = \partial_0 + \partial_1 + \partial_{\geq 2}$ such that $\partial_0\colon V \to V$ is the boundary operator of cells, $\partial_1\colon V \to \widehat{\mathbb{L}}^2V$ comes from a cellular approximation of the diagonal and $\partial_{\geq 2}(V) \subset \widehat{\mathbb{L}}^{>2}V$. In the case of the interval I, with two 0-cells and one 1-cell, this model is of the shape $(\widehat{\mathbb{L}}(a,b,x),\partial)$, with $|a| = |b| = -1$, $|x| = 0$ and $\partial_0 x = b - a$. More details are given in Section 1. We call this model *the inductive model of the interval.*

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In [2], R. Lawrence and D. Sullivan prove the existence of a completed differential Lie algebra $(\widehat{\mathbb{L}}(a,b,x),\partial)$, such that $\partial a = -(1/2)[a,a]$, $\partial b = -(1/2)[b,b]$ and

$$\partial x = \mathrm{ad}_x(b) + \sum_{i=0}^{\infty} \frac{B_i}{i!}(\mathrm{ad}_x)^i(b-a),$$

where the B_i are the Bernoulli numbers. This construction comes from an analysis of the flow generated by x which moves from a to b in unit time, with a and b being flat. We call it *the geometric model*.

In the two papers, [6] and [2], it is conjectured that these two models are the same. We prove this conjecture here.

Main Theorem . *The inductive and the geometric models of the interval are the same.*

The proof consists in two preliminary steps. In Section 1, we revisite the inductive construction taking into account the particular case of the interval. A second ingredient comes from the study of some derivations of $\mathbb{L}(x,\beta)$, $|x|=0$ and $|\beta|=-1$, done in Section 2. Finally, Section 3 contains the proof of the conjecture using, among other tools, the Euler formula which characterizes the Bernoulli numbers. As a bonus, our proof generates other relations between the Bernoulli numbers. In consideration of the litterature on the subject, they are certainly well-known but we do not have any reference for them.

Finally, one may observe as in [2] that this differential (completed) Lie algebra is similar to the Quillen's model, introduced in [3] for the study of the rational homotopy types of CW-complexes with one 0-cell and no 1-cell, in contrast with the Sullivan approach ([4]) which authorizes nilpotent spaces of finite type. Through the dictionary between infinity cocommutative coassociative coalgebra structure and differential of a free Lie algebra, this model is an explicit construction of the infinity cocommutative coassociative coalgebra structure on the rational chains of the interval.

1. Sullivan's inductive construction

In this section, we adapt Sullivan's proof of [6] to the particular case of the interval, and prove the next property.

Proposition 1. *The Sullivan's inductive model $(\widehat{\mathbb{L}}(a,b,x),\partial)$ of the interval admits a differential of the form $\partial a = -\frac{1}{2}[a,a]$, $\partial b = -\frac{1}{2}[b,b]$ and*

$$\partial x \;=\; ad_x(b) \;+\; \sum_{i=0}^{\infty} \frac{\lambda_i}{i!} ad_x^i(\beta),$$

with $\beta = b-a$, $\lambda_i \in \mathbb{Q}$ and $\lambda_{2k+1}=0$, $k\geq 1$.

Proof. We proceed by induction, supposing that a derivation $\partial_{\leq n}$ on $\mathbb{L}(a,b,x)$ has been constructed such that

$$\partial_{\leq n} a \;=\; -\frac{1}{2}[a,a]\,,\; \partial_{\leq n} b \;=\; -\frac{1}{2}[b,b]\,,\; \partial_{\leq n} x \;=\; ad_x(b) \;+\; \sum_{i=0}^{n} \frac{\lambda_i}{i!} ad_x^i(\beta),$$

with $\lambda_i \in \mathbb{Q}$ satisfying

$$\mathrm{Im}\,\partial_{\leq n}^2 \subset \mathbb{L}^{\geq n+2}(x,\beta), \text{ for } n\geq 1,$$

and

$$\operatorname{Im} \partial_n \subset \mathbb{L}(x,\beta), \text{ for } n \geq 2.$$

The induction starts at $n = 1$. We first verify that case. Consider the derivations ∂_0 and ∂_1 on $\mathbb{L}(a, b, x)$, given by $\partial_0 a = \partial_0 b = 0$, $\partial_0 x = \beta$, and

$$\partial_1 a = -\frac{1}{2}[a, a]\,,\ \partial_1 b = -\frac{1}{2}[b, b]\,,\ \partial_1 x = \frac{1}{2} ad_x(b + a).$$

Clearly $\partial_0^2 = \partial_0\partial_1 + \partial_1\partial_0 = 0$ and we proceed with the computation of ∂_1^2. The Jacobi identity implies that triple brackets of the form $[\alpha, [\alpha, \alpha]]$ are all zero. Hence $\partial_1^2 a = \partial_1^2 b = 0$. In contrast, $\partial_1^2 x \neq 0$. A computation using the Jacobi identity shows that

$$\partial_1^2 x \,=\, \frac{1}{2}\,\partial_1(ad_x(b + a)) \,=\, -\frac{1}{8} ad_x[\beta, \beta] \in \mathbb{L}^3(x, \beta).$$

Suppose now that $\partial_{\leq n}$ has been constructed as before. The set of derivations on a Lie algebra carries a natural Lie structure. Hence, we must have

$$[\partial_{\leq n}, [\partial_{\leq n}, \partial_{\leq n}]] = 0. \tag{1}$$

Moreover, since $|\partial_{\leq n}| = -1$, we have $[\partial_{\leq n}, \partial_{\leq n}] = 2\,\partial_{\leq n}^2$ and $[\partial_{\leq n}, \partial_{\leq n}](x) \in \mathbb{L}^{\geq n+2}(x, \beta)$ by the induction hypothesis. Hence the component in $\mathbb{L}^{n+2}(x, \beta)$ of the triple bracket (1) evaluated at x is given by

$$0 = [\partial_0, \sum_{i+j=n+1} \partial_i \circ \partial_j](x) = \partial_0(\sum_{i+j=n+1} \partial_i \circ \partial_j)(x),$$

The element $\displaystyle\sum_{i+j=n+1} \Big(\partial_i \circ \partial_j\Big)(x) \in \mathbb{L}^{n+2}(x, \beta)$ being a ∂_0-cycle of total degree -2 and the dgL $(\mathbb{L}(x, \beta), \partial_0)$ being acyclic, one can find an element $\gamma \in \mathbb{L}^{n+2}(x, \beta)$ of total degree -1 such that

$$\partial_0 \gamma = \sum_{i+j=n+1} \Big(\partial_i \circ \partial_j\Big)(x).$$

The subspace of $\mathbb{L}^{n+2}(x, \beta)$ in total degree -1 being generated by $ad_x^{n+1}(\beta)$, γ must be some (rational) multiple η of that element. Hence we can extend $\partial_{\leq n}(x)$ with

$$\partial_{n+1}(x) = \frac{\lambda_{n+1}}{(n + 1)!} \mathrm{ad}_x^{n+1}(\beta)$$

by choosing $\lambda_{n+1} = -(n + 1)! \cdot \eta$. By construction one has the inclusion

$$\operatorname{Im} \partial_{\leq n+1}^2 \subset \mathbb{L}^{\geq n+3}(x, \beta), \ \text{ and } \operatorname{Im} \partial_{n+1} \subset \mathbb{L}(x, \beta).$$

This completes the inductive step. Let us show that one can choose $\lambda_{2k+1} = 0$ if $k \geq 1$. First, notice that the restriction of ∂_1 to $\mathbb{L}(x, \beta)$ satisfies

$$\partial_1|_{\mathbb{L}(x,\beta)} = -\frac{1}{2} ad_{a+b}.$$

By definition, it is true on x while on β we have

$$\partial_1 \beta = -\frac{1}{2}[b, b] + \frac{1}{2}[a, a] = -\frac{1}{2}[b + a, b - a] = -\frac{1}{2} ad_{a+b}\beta.$$

Hence for all k, $1 < k$, we have

$$\partial_1 \circ \partial_k = -\frac{1}{2} ad_{a+b} \circ \partial_k,$$

since by hypothesis $\partial_k(\mathbb{L}(a,b,x)) \subset \mathbb{L}(x,\beta)$. Moreover, we have $\partial_k \circ \partial_1(a) = \partial_k \circ \partial_1(b) = 0$ and $\partial_k \circ \partial_1(x) = -\frac{1}{2}\partial_k ad_{a+b}(x) = \frac{1}{2}ad_{a+b}\partial_k(x)$, i.e.,

$$\partial_k \circ \partial_1 = \frac{1}{2}ad_{a+b} \circ \partial_k.$$

Suppose that $\lambda_{2k+1} = 0$ for $1 < 2k+1 < n$ with n even. We can identify the $(n+2)$-bracket length of $\partial^2_{\leq n}(x)$ with $(\partial_1 \circ \partial_n + \partial_n \circ \partial_1)(x)$, which is trivial as we have seen. Hence, we can extend $\partial_{\leq n}$ to $\partial_{\leq n+1}$ by choosing $\lambda_{n+1} = 0$. This ends the proof. □

2. Derivations of $\mathbb{L}(x,\beta)$

Consider a graded Lie algebra $L = \mathbb{L}(a,b,x)$ generated by two elements a and b of degree -1, and one element x of degree 0. Let $\beta = b - a$ and $\gamma \in \mathbb{L}(x,\beta)$ be an element of degree -1. We set

$$\mu_{n,k}(\gamma) = \mathrm{ad}_x^{n-2k}([\mathrm{ad}_x^k(\gamma), \mathrm{ad}_x^k(\gamma)]) \in \mathbb{L}^{n+2}(x,\beta).$$

The elements $\mu_{n,k}(\beta)$ of L play a crucial role in the inductive model. To give a more explicit expression of this model, we study the composition of some adjoint representations in $\mathbb{L}(x,\beta)$. Let $v_{n,k}$ be the rational numbers defined by

$$\begin{aligned} v_{n,k} &= v_{n-1,k} - v_{n-2,k-1}, \\ v_{0,0} &= 1,\ v_{1,0} = 1/2, \\ v_{i,j} &= 0 \text{ if } j < 0 \text{ or if } j > [i/2]. \end{aligned}$$

Proposition 2. *Let $\gamma \in \mathbb{L}(x,\beta)$ be an element of degree -1. For any $p \geq 0$ and $q \geq 0$, we have*

$$\mathrm{ad}_{\mathrm{ad}_x^p(\gamma)} \circ \mathrm{ad}_x^q(\gamma) = \sum_k v_{|p-q|,k}\, \mu_{p+q,\mathrm{Inf}(p,q)+k}(\gamma).$$

Proof. This property is clearly satisfied for $p = 0$ and $q = 0$. Suppose it is true for $p = 0$ and $0 \leq q \leq n-1$. The Jacobi identity and the induction hypothesis imply:

$$\begin{aligned} \mathrm{ad}_\gamma \circ \mathrm{ad}_x^n(\gamma) &= \mathrm{ad}_x([\mathrm{ad}_x^{n-1}(\gamma), \gamma]) + [\mathrm{ad}_x^{n-1}(\gamma), [\gamma, x]] \\ &= \mathrm{ad}_x \circ \mathrm{ad}_\gamma \circ \mathrm{ad}_x^{n-1}(\gamma) - \mathrm{ad}_{[x,\gamma]} \circ \mathrm{ad}_x^{n-2}([x,\gamma]) \\ &= \mathrm{ad}_x(\sum_k v_{n-1,k}\, \mu_{n-1,k}(\gamma)) - \sum_k v_{n-2,k}\, \mu_{n-2,k}([x,\gamma]) \\ &= \sum_k (v_{n-1,k} - v_{n-2,k-1})\ \mu_{n,k}(\gamma). \end{aligned}$$

Thus the formula is proved for $p = 0$. We distinguish now the two cases, $p \geq q$ and $q \geq p$. Let $i \geq 0$ and $j \geq 0$.

- If $p = j$, $q = i + j$, then from

  $$\mathrm{ad}_{\mathrm{ad}_x^j(\gamma)} \circ \mathrm{ad}_x^{j+i}(\gamma) = \mathrm{ad}_{\mathrm{ad}_x^j(\gamma)} \circ \mathrm{ad}_x^i(\mathrm{ad}_x^j(\gamma))$$

  and from our first step applied to $\mathrm{ad}_x^j(\gamma)$, we get

  $$\mathrm{ad}_{\mathrm{ad}_x^j(\gamma)} \circ \mathrm{ad}_x^{j+i}(\gamma) = \sum_k v_{i,k}\, \mu_{i+2j,j+k}(\gamma).$$

- If $p=i+j$ and $q=j$, then using Jacobi identity and the first step, we have

$$\mathrm{ad}_{\mathrm{ad}_x^i(\gamma)}(\gamma) = [\mathrm{ad}_x^i(\gamma),\gamma] = \mathrm{ad}_\gamma \circ \mathrm{ad}_x^i(\gamma) = \sum_k v_{n,k}\,\mu_{n,k}(\gamma).$$

Now replacing γ by $\mathrm{ad}_x^i(\gamma)$ to get

$$\mathrm{ad}_{\mathrm{ad}_x^{i+j}(\gamma)} \circ \mathrm{ad}_x^i(\gamma) = \sum_k v_{j,k}\,\mu_{j+2i,k+i}(\gamma).$$

□

Corollary 3. *The elements $\mu_{n,k}(\beta)\in\mathbb{L}(x,\beta)$ are linearly independent.*

Proof. Recall first a well-known property concerning free Lie algebras (see [5, Proposition VI.2.(7) Page 139] for instance). Let V and W be rational vector spaces. Then the kernel of the canonical projection $\mathbb{L}(V\oplus W)\to\mathbb{L}(V)$ is the free Lie algebra on $T(V)\otimes W$, the canonical injection $\mathbb{L}(T(V)\otimes W)\to\mathbb{L}(V\oplus W)$ corresponding to adjunctions. More precisely, in our case, the kernel of the projection $\mathbb{L}(x,\beta)\to\mathbb{L}(x)$ is $\mathbb{L}(T(x)\otimes\beta)$ and the canonical inclusion $j\colon \mathbb{L}(T(x)\otimes\beta)\to\mathbb{L}(x,\beta)$ is defined by $j(x^n\otimes\beta)=\mathrm{ad}_x^n(\beta)$. We repeat this process: the kernel of the projection $\mathbb{L}(T(x)\otimes\beta)\to\mathbb{L}(\beta)$ is $\mathbb{L}(T(\beta)\otimes T^+(x)\otimes\beta)$.

Let $n>0$ be fixed and $L\langle n\rangle$ be the vector subspace of $\mathbb{L}(x,\beta)$ formed of brackets with exactly n letters x and twice the letter β. With the identification coming from the canonical inclusions, $L\langle n\rangle$ is a subspace of $\mathbb{L}(T(\beta)\otimes T^+(x)\otimes\beta)$ spanned by the generator $\beta\otimes x^n\otimes\beta$ and the brackets $[x^i\otimes\beta, x^j\otimes\beta]$, with $i+j=n$. If we impose $i\geq j$, these elements are a basis of $L\langle n\rangle$. Suppose that $n=2p$ for sake of simplicity, the argument being similar for n odd. The previous considerations show that $\mathbb{L}\langle n\rangle$ is of dimension $p+1$. Now Proposition 2 implies that the $(p+1)$ elements $\mu_{n,k}(\beta)$ span $\mathbb{L}\langle n\rangle$. Thus, they are a basis of $\mathbb{L}\langle n\rangle$. □

We denote by θ_n the derivation of $\mathbb{L}(x,\beta)$, defined by $\theta_n(x)=ad_x^n(\beta)$ and $\theta_n(\beta)=0$, for any $n\geq 0$.

Corollary 4. *The image of the composition $\theta_p\circ\theta_q$ is contained in the linear span of the $\mu_{p+q-1,k}(\beta)$. More precisely, if $p\geq q$, then we have*

$$\theta_p\circ\theta_q(x)=\sum_k\left(\sum_{i=0}^{q-1} v_{p-i,k-i}\right)\mu_{p+q-1,k}(\beta),$$

and when $p<q$, the formula becomes

$$\theta_p\circ\theta_q(x)=\sum_k\left(\sum_{i=0}^{q-p-1} v_{q-p-1-i,k-p}+\sum_{i=1}^{p} v_{i,k-p+i}\right)\mu_{p+q-1,k}(\beta).$$

Proof. Clearly $\theta_p\circ\theta_0\equiv 0$. So we assume $p\geq 0$ and $q\geq 1$.

$$\begin{aligned}\theta_p\circ\theta_q(x) &= [\theta_p(x),\theta_{q-1}(x)]+[x,\theta_p\circ\theta_{q-1}(x)],\\ &= \mathrm{ad}_{\mathrm{ad}_x^p(\beta)}\circ\mathrm{ad}_x^{q-1}(\beta)+\mathrm{ad}_x\circ\theta_p\circ\theta_{q-1}(x).\end{aligned}$$

Hence, with an iteration on the second term and Proposition 2, one gets

$$\begin{aligned}
\theta_p \circ \theta_q(x) &= \sum_{i=0}^{q-1} \mathrm{ad}_x^i \circ \mathrm{ad}_{\mathrm{ad}_x^p(\beta)} \circ \mathrm{ad}_x^{q-i-1}(\beta) \\
&= \sum_{i=0}^{q-1} \mathrm{ad}_x^i \circ \sum_t v_{|p-q+i+1|,t}\, \mu_{p+q-i-1,\mathrm{Inf}(p,q-i-1)+t}(\beta) \\
&= \sum_{i=0}^{q-1} \sum_t v_{|p-q+i+1|,t}\, \mu_{p+q-1,\mathrm{Inf}(p,q-i-1)+t}(\beta).
\end{aligned}$$

If $p \geq q$, this formula simplifies in

$$\begin{aligned}
\theta_p \circ \theta_q(x) &= \sum_{i=0}^{q-1} \sum_t v_{p-q+i+1,t}\, \mu_{p+q-1,q-i-1+t}(\beta) \\
&= \sum_k \sum_{j=0}^{q-1} v_{p-j,k-j}\, \mu_{p+q-1,k}(\beta),
\end{aligned}$$

with $k = q+t-i-1$ and $j = q-1-i$. If $p < q$, we have to cut the formula in two parts

$$\begin{aligned}
\theta_p \circ \theta_q(x) &= \sum_t \sum_{i=0}^{q-p-1} v_{q-p-1-i,t}\, \mu_{p+q-1,p+t}(\beta) \\
&\qquad + \sum_t \sum_{i=q-p}^{q-1} v_{p-q+1+i,t}\, \mu_{p+q-1,q-i-1+t}(\beta).
\end{aligned}$$

The result follows by a simple but tedious re-indexing, as we did in the first case. □

3. Proof of the Main Theorem

There is no universally accepted convention for the Bernoulli numbers. Here we choose the following one:

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}\, z^k = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\, z^{2k}.$$

All the Bernoulli numbers are rational, with $B_1 = -(1/2)$, $B_{2k+1} = 0$ if $k \geq 1$,

$$B_0 = 1,\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730}.$$

Bernoulli numbers verify several induction formulae. Amongst them we recall the Euler formula, i.e.,

$$-n\, B_n = \sum_{k=1}^{n} \binom{n}{k} B_k\, B_{n-k} + n\, B_{n-1}.$$

When n is even with $n > 2$, the formula reduces to

$$-\frac{(n+1)\, B_n}{n!} = \sum_{k=2}^{n-2} \frac{B_k}{k!}\, \frac{B_{n-k}}{(n-k)!}. \tag{2}$$

Theorem 5. *The coefficients of the differential in Proposition 1 are given by the Bernouilli numbers, i.e.,*

$$\lambda_i = B_i,$$

for all $i \geq 0$.

Proof. We check easily that $\lambda_0 = B_0 = 1$ and $\lambda_1 = B_1 = -\frac{1}{2}$. Moreover, the construction guarantees that $\lambda_{2k+1} = B_{2k+1} = 0$ for $k \geq 1$. Recall that within the proof of Proposition 1, we have shown that $\partial_1^2(x) = -\frac{1}{8}ad_x[\beta,\beta]$. A simple computation shows that $\partial_0([x,[x,\beta]]) = \frac{3}{2}ad_x[\beta,\beta]$. That implies that we can choose $\lambda_2 = B_2 = \frac{1}{6}$.

Now let us assume $n > 2$ and even. Observe from Corollary 4 that, for $p, q \geq 2$ such that $p+q = n$, the rational coefficient of $\mu_{n-1,0}(\beta)$ in the expression of $\theta_p \circ \theta_q(x)$ is given by

$$v_{p,0} = \frac{1}{2}.$$

Hence, on one hand, the coefficient of $\mu_{n-1,0}(\beta)$ in the equation

$$\sum_{k=2}^{n-2} \partial_k \circ \partial_{n-k}(x) \; = \sum_{k=2}^{n-2} \frac{\lambda_k}{k!} \frac{\lambda_{n-k}}{(n-k)!} \theta_k \circ \theta_{n-k}(x)$$

is

$$\frac{1}{2} \sum_{k=2}^{n-2} \frac{\lambda_k}{k!} \frac{\lambda_{n-k}}{(n-k)!}.$$

On the other hand, one has by Corollary 4 that the rational coefficient of $\mu_{n-1,0}(\beta)$ in the expression of $\theta_0 \circ \theta_n(x)$ is

$$1 + \sum_{k=1}^{n-1} v_{k,0} = 1 + \frac{n-1}{2} = \frac{n+1}{2}.$$

Since by construction we have

$$\partial_0 \circ \partial_n \; + \sum_{i=2}^{n-2} \partial_i \circ \partial_{n-i} = 0,$$

the following relation must be satisfied, i.e.,

$$\sum_{k=2}^{n-2} \frac{\lambda_k}{k!} \frac{\lambda_{n-k}}{(n-k)!} = -(n+1)\frac{\lambda_n}{n!},$$

which is no other than the Euler equation characterizing the Bernouilli numbers. The result follows.

□

4. Other Euler type relations between Bernouilli numbers

The proof generates other relations amongst the Bernouilli numbers. Indeed, when n is even, we can deduce such relations by projecting

$$\partial_0 \circ \partial_n \; + \sum_{i=2}^{n-2} \partial_i \circ \partial_{n-i} = 0,$$

onto any $\mu_{n-1,k}(\beta)$.

Proposition 6. *For any fixed k, we have*

$$
\begin{aligned}
-\frac{B_n}{n!}\left(\sum_{l=0}^{n-1} v_{l,k}\right) &= \sum_{\substack{2\le i<\frac{n}{2}\\ k<i}} \frac{B_i}{i!}\frac{B_{n-i}}{(n-i)!}\left(\sum_{l=0}^{i} v_{l,k-i+l}\right)\\
&\quad + \sum_{\substack{2\le i<\frac{n}{2}\\ k\ge i}} \frac{B_i}{i!}\frac{B_{n-i}}{(n-i)!}\left(\sum_{l=0}^{n-2i-1} v_{l,k-i}\right)\\
&\quad + \sum_{i=\frac{n}{2}}^{n-2} \frac{B_i}{i!}\frac{B_{n-i}}{(n-i)!}\left(\sum_{l=0}^{n-i-1} v_{i-l,k-l}\right).
\end{aligned}
$$

Proof. We project $\partial_0\circ\partial_n+\sum_{i=2}^{n-2}\partial_i\circ\partial_{n-i}=0$, onto $\mu_{n-1,k}(\beta)$. On one hand, the coefficient of $\mu_{n-1,k}(\beta)$ in the expression of $\partial_0\circ\partial_n(x)$ is

$$\frac{B_n}{n!}\left(\sum_{l=0}^{n-1} v_{l,k}\right).$$

While on the other hand, when $p+q=n$ and $p<q$, the coefficient of $\mu_{n-1,k}(\beta)$ in the expression of $\partial_p\circ\partial_q(x)$ is given by

$$\frac{B_p}{p!}\frac{B_q}{q!}\begin{cases}\left(\sum_{l=0}^{p} v_{l,k-p+l}\right), & k<p\\ \left(\sum_{l=0}^{q-p-1} v_{l,k-p}\right), & k\ge p\end{cases},$$

and when $p\ge q$ it is given by

$$\frac{B_p}{p!}\frac{B_q}{q!}\left(\sum_{l=0}^{q-1} v_{p-l,k-l}\right).$$

□

Remark 7. The previous relations cannot be directly reduced to Euler equation. For instance, when $n=8$ and $k=2$, one gets the relation

$$\frac{9}{2}\frac{B_2}{2!}\frac{B_6}{6!}=-15\frac{B_8}{8!}$$

which differs from the Euler equation simply by the fact that the coefficient of $\dfrac{B_4^2}{(4!)^2}$ is zero. If one considers the case $n=10$ and $k=2$, we get the relation

$$\frac{5}{2}\frac{B_4}{4!}\frac{B_6}{6!}+10\frac{B_2}{2!}\frac{B_8}{8!}=-\frac{77}{2}\frac{B_{10}}{10!},$$

in which no terms are missing but still differs from Euler's relation because the coefficients on the left hand side are not equal.

Finally, we observe that the numbers $v_{n,k}$ can be explicitely determined.

Proposition 8. *The sequence $v_{n,k}$ satisfy the following properties:*

$$2\, v_{n,k} = (-1)^k \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right), \ \textit{and} \ \sum_{k=0}^{n} v_{n+k,k} = 0.$$

Proof. Let $f_0(n) = n$. We set

$$f_{k+1}(n) = \sum_{i=1}^{n} f_k(n).$$

Recall from [1, Page 134] that

$$f_{k+1}(n) = \binom{n+k-1}{k} f_0(1) + \cdots + \binom{k}{k} f_0(n) = \binom{n+k+1}{k+2}. \tag{3}$$

In the above expression of $v_{n,k}$, the sign is clear thus we have only to study the absolute value. We introduce $\sigma_{n,k} = \sum_{i=0}^{n} v_{i,k}$. If we add, from 0 to n, the defining relation $v_{n,k} = v_{n-1,k} - v_{n-2,k-1}$, we get

$$v_{n,k} = -\sum_{i=0}^{n-2} v_{i,k-1} = -\sigma_{n-2,k-1}, \tag{4}$$

and, by adding these relations from 0 to k, we get

$$\sigma_{n,k} = -\sum_{i=0}^{n-2} \sigma_{i,k-1}. \tag{5}$$

We observe that $2|\sigma_{n,0}| = n + 2 = f_0(n) + 2$ and

$$\begin{aligned} 2\,|\sigma_{n,1}| &= \sum_{i=0}^{n-2} |\sigma_{i,0}| = \sum_{i=0}^{n-2} (f_0(i) + 2) \\ &= f_1(n-2) + 2\, f_0(n-1). \end{aligned}$$

More generally, an induction on k using formula (5) gives

$$2\,|\sigma_{n,k}| = f_k(n-2k) + 2\, f_{k-1}(n-2k+1).$$

Formulae (3), (4) and basic properties of binomial coefficients give the result. □

For convenience, we supply the first values of the $v_{n,k}$.

n	$v_{n,0}$	$v_{n,1}$	$v_{n,2}$	$v_{n,3}$
0	1			
1	1/2			
2	1/2	-1		
3	1/2	-3/2		
4	1/2	-2	1	
5	1/2	-5/2	5/2	
6	1/2	-3	9/2	-1
7	1/2	-7/2	7	-7/2

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